— Exercises —

1. **Computation of a Jacobian matrix and use of chain rule.**

   (a) Compute $Jf(r, \theta)$ and its determinant for

   $$\begin{aligned} f: \ \mathbb{R}^2 &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta, r\sin\theta). \end{aligned}$$

   When is $df(r, \theta)$ invertible?

   (b) Define

   $$\begin{aligned} g: \ \mathbb{R}^2 &\to \mathbb{R}^3 \\ (x, y) &\mapsto (1, xy, x^2y^2). \end{aligned}$$

   What is the derivative of $g \circ f$ at $(r = 2, \theta = \pi/4)$?

2. **Special cases of chain rule.** Let $f : \mathbb{R}^p \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^q$ be two differentiable functions and $c \in \mathbb{R}^p$.

   (a) If $n = q = 1$, express $\nabla(g \circ f)(c)$ in terms of g' and ∇f.
   Hint: use that $\forall t, < \nabla(g \circ f)(c), t >= d(f \circ g)(c).t$ *and the chain rule.*

   (b) If $p = q = 1$, express $df(c).t$ in terms of $f'(c)$ and $(g \circ f)'(c)$ in terms of ∇g and $f'(c)$, where $f'(c) := \big(f_1'(c), \dots, f_n'(c)\big)$.

— Problems —

3. **Norm on the set of linear maps $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.** For a linear map A with matrix $(a_{ij})_{i,j}$, define $\|A\| := \sqrt{\sum_{i,j} a_{ij}^2}$. This is a norm on the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Show that for all A and B, $\|AB\| \leq \|A\| \, \|B\|$. Show that for fixed B, the map $A \mapsto AB$ is continuous.

4. **Derivative of "taking the inverse".** Let $GL_n(\mathbb{R}) \subset \mathbb{R}^{n \times n}$ denote the set of invertible $n \times n$ matrices. We use the norm above, i.e., we identify $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n^2}$ with its Euclidean norm.

   (a) Justify that $GL_n(\mathbb{R})$ is open.

   (b) Let $I_n \in GL_n(\mathbb{R})$ be the identity matrix. Show that if T is a matrix with small enough coefficients,

   $$(I_n + T)^{-1} = \sum_{k=0\dots\infty} (-1)^k T^k.$$

   (c) Let

   $$\begin{aligned} \psi : GL_n(\mathbb{R}) &\to GL_n(\mathbb{R}) \\ A &\mapsto A^{-1} \end{aligned} \ .$$

   Show that ψ is differentiable at I_n and compute its derivative.

   (d) Let $A \in GL_n(\mathbb{R})$. Show that ψ is differentiable at A and compute its derivative. Compare with $(\frac{1}{x})' = -\frac{1}{x^2}$!